J. Fluid Mech. (1967), *vol.* 27, *part* 1, *pp.* 65–79
Printed in Great Britain

Hydromagnetic stability of dissipative flow between rotating permeable cylinders

Part 1. Stationary critical modes

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(Received 16 September 1965, and in revised form 12 April 1966)

The theory of stability of the flow of a viscous, electrically conducting fluid between rotating cylinders in the presence of an axial magnetic field is extended to the case where the cylinders are permeable and the primary flow includes a radial component. Numerical results pertaining to the stationary axially symmetric modes are presented, and the asymptotic stability behaviour for large values of the radial Reynolds number is derived.

1. Introduction

Recent concepts of gaseous-core nuclear reactors (Kerrebrock & Meghreblian 1961) and plasma power-generating devices (Lewellen 1960) require stabilized laminar motion of conducting fluids similar to that between permeable, rotating cylindrical walls with a large gap width. This paper considers a wide-gap hydromagnetic-stability analysis of such a flow under the influence of an axially applied magnetic field. The fluid is assumed to be of uniform density, with finite viscosity and conductivity; the walls are assumed to be perfectly conducting.

Chandrasekhar (1953, 1961), Niblett (1958), Kurzweg (1963), Roberts (1964), Chang & Sartory (1965*a*) have discussed in detail the stability results for dissipative flow between non-permeable, rotating cylinders subject to various types of magnetic conditions. Hazlehurst (1963) considered the problem of combined rotating and radial flow in the non-magnetic inviscid limit. In this paper, we shall consider the effect of radial flow on the criterion of hydromagnetic stability. The asymptotic behaviour of the criterion at high values of the radial Reynolds number will be deduced. Numerical results for perfectly conducting walls will also be presented.

The results reported here apply only to stationary axisymmetric critical modes. The possible effect of oscillatory critical modes is also mentioned in § 5. The stability of oscillatory modes will be considered in more detail in part 2 of this series.

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2. Formulation of the problem

2.1. *The stationary solution*

It is easily verified that the basic hydromagnetic equations admit a stationary flow of the form

$$(u, v, w) = (N'/r, L'/r + M'r^{R_r+1}, 0), \tag{2.1}$$

with a magnetic field

$$(B_r, B_\theta, B_z) = (0, 0, B_0' + B_1' r^{P_m R_r}), \tag{2.2}$$

where

$$P_m \text{ (magnetic Prandtl number)} = \mu_0 \sigma \nu, \tag{2.3}$$

$$R_r \text{ (radial Reynolds number)} = N'/\nu, \tag{2.4}$$

(u, v, w), (B_r, B_θ, B_z) are the cylindrical components of the velocity vector and magnetic induction field, respectively, r is the radial co-ordinate, ν is the kinematic viscosity, μ_0 is the magnetic permeability, σ is the electrical conductivity, L', M', N', B_0', B_1' are constants, and the rationalized M.K.S. system of units is used.

Equations (2.1) and (2.2) indicate that the radial-flow component introduces a radial variation of the axially applied magnetic field. The constants B_0' and B_1' are to be determined by the magnetic boundary conditions. For most fluids of practical interest, the magnetic Prandtl number is small ($\gtrsim 10^{-6}$). Setting $P_m = 0$ in (2.2), we find that the admissible stationary magnetic field becomes uniform. We note that this approximation is valid only if $P_m|R_r| \ll 1$. Under the assumption of $P_m|R_r| \ll 1$, it is easily verified that a uniform magnetic field is admissible for perfectly conducting magnetic boundary conditions at the walls and that the magnetic field has the same strength as the externally applied field.

2.2. *Normal mode equations*

In our analysis, we shall consider the admissible stationary flow to be given by (2.1) with the magnetic field given by $(0, 0, B_0)$, the uniform applied axial magnetic field. We consider in this paper the linear stability problem. Allowing the stationary solution to deviate slightly, and resolving the perturbations into axisymmetric normal modes whose time t and axial dependence z are of the form $\exp(ipt + ikz)$, we obtain, by inserting the perturbed solution into the basic hydromagnetic equations and dropping higher-order terms, the following set of equations in reduced dimensionless form:

$$\left.\begin{aligned}
&\{(1/R)(DD_+ - \beta_2^2) - (N/x)D_- - i\alpha\}(DD_+ - \beta_2^2)y_1 + (\beta_2/A_2)(DD_+ - \beta_2^2)y_3 \\
&\qquad\qquad = 2\beta_2^2\left\{(L/x^2) + \frac{2M}{R_r+2}x^{R_r}\right\}y_2, \\
&\{(1/R)(DD_+ - \beta_2^2) - (N/x)D_+ - i\alpha\}y_2 = 2Mx^{R_r}y_1 - (\beta_2/A_2)y_4, \\
&\{(1/R_m)(DD_+ - \beta_2^2) - (N/x)D_+ - i\alpha\}y_3 = \beta_2 y_1, \\
&\{(1/R_m)(DD_+ - \beta_2^2) - (N/x)D_- - i\alpha\}y_4 = \beta_2 y_2 + 2\left\{(L/x^2) - \frac{MR_r}{R_r+2}x^{R_r}\right\}y_3,
\end{aligned}\right\} \tag{2.5}$$

where $y_1(x)$, $y_2(x)$ and $y_3(x)$, $y_4(x)$ are the dimensionless amplitudes of the normal mode perturbations of the radial r- and transverse θ-components of the velocity vector and the magnetic induction vector, respectively. The remaining symbols in (2.5) are defined as follows:

$$\left.\begin{array}{c} \alpha = pr_2/V_0, \quad \beta_2 = kr_2, \\ L = L'/(r_2 V_0), \quad M = M' r_2^{R_r+1}/V_0, \quad N = N'/(r_2 V_0), \\ x = r/r_2, \qquad D = d/dx, \qquad D_{\pm} = D \pm 1/x, \\ A \text{ (Alfvén number)} = V_0 \sqrt{(\rho \mu_0)}/B_0, \\ R \text{ (Reynolds number)} = V_0 r_2/\nu, \\ R_m \text{ (magnetic Reynolds number)} = V_0 r_2 \mu_0 \sigma, \end{array}\right\} \tag{2.6}$$

where V_0 is a characteristic speed, ρ is the fluid density, and r_2 is the radius of the outer cylinder.

Equations (2.5), when considered with a set of homogeneous velocity and magnetic boundary conditions, from the basic eigenvalue problem. We wish to decide if, for real β_2, it is possible to choose values of B_0, such that only the eigensolutions with $\operatorname{Im} \alpha > 0$ are admissible, restricting the normal mode perturbations to be decaying in nature.

2.3. *Boundary conditions*

We assume that the radial velocity through the walls is set externally in such a way that it is unaffected by the perturbations within the fluid. Thus, we must have

$$y_1 = 0 \quad \text{at} \quad x = \kappa, 1, \tag{2.7}$$

where $\kappa = r_1/r_2$; r_1 being the radius of the inner wall. In reality, the velocity perturbations may penetrate into the permeable walls. If the walls are sufficiently thick and made of a material with a sufficiently small permeability to the passage of fluid, however, the magnitude of the disturbance in the walls can be made arbitrarily small so that (2.7) is approximately satisfied. The primary radial flow can still be maintained by applying a sufficient pressure differential.

From the non-slip condition, the incompressibility assumption, and (2.7), we also deduce that

$$Dy_1, y_2 = 0 \quad \text{at} \quad x = \kappa, 1. \tag{2.8}$$

For perfectly conducting walls, the axial component of the electric field E_z must vanish at the walls. Applying Ohm's law and utilizing one of Maxwell's equations with the displacement current neglected according to the basic hydromagnetic assumption, we find that

$$\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} - \mu_0 \sigma \mu B_\theta = 0 \quad \text{at} \quad r = r_1, r_2. \tag{2.9}$$

In terms of the dimensionless perturbation amplitude function, we have

$$(D_+ - P_m R_r/x) y_4 = 0 \quad \text{at} \quad x = \kappa, 1. \tag{2.10}$$

In our analysis, the product $P_m |R_r|$ is assumed to be small. Setting $P_m |R_r| = 0$, the required velocity and magnetic boundary conditions become

$$y_1, Dy_1, y_2, D_+ y_4 = 0 \quad \text{at} \quad x = \kappa, 1. \tag{2.11}$$

Because of a reduction of the order of the equations (2.5) for $P_m = 0$, which is assumed in our analysis, (2.11) suffice as the required boundary conditions.

2.4. *Reduced normal mode equations*

We work with the $P_m = 0$ approximation. Assuming that the critical mode is due to a secondary stationary flow, we obtain from (2.5) the following set of coupled equations for the hydromagnetic stability of the flow between permeable walls

$$\left.\begin{aligned}&[\{DD_+ - \beta_2^2 - (R_r/x)D_-\}\{DD_+ - \beta_2^2\} + \beta_2^2 H^2]W_1\\ &\qquad + \beta_2^2 T\{(1/x^2) + \lambda x^{R_r}\}(DD_+ - \beta_2^2)W_4 = 0,\\ &[\{DD_+ - \beta_2^2 - (R_r/x)D_+\}\{DD_+ - \beta_2^2\} + \beta_2^2 H^2]W_4 - x^{R_r}W_1 = 0,\end{aligned}\right\} \tag{2.12}$$

where

$$\left.\begin{aligned}&(W_1, W_4) = (2MRy_1, y_4/\beta_2 R_m),\\ &T \text{ (Taylor number)} = -4LMR^2,\\ &Q_2 = RR_m/A^2 = \sigma B_0^2 r_2^2/\mu,\\ &\lambda = \frac{2M/L}{R_r + 2}.\end{aligned}\right\} \tag{2.13}$$

The boundary conditions corresponding to (2.12) are

$$W_1,\ DW_1,\ D_+W_4,\ (DD_+ - \beta_2^2)W_4 = 0 \quad \text{at} \quad x = \kappa, 1. \tag{2.14}$$

We note that in the present formulation, the boundary condition for B_r is not required.

3. Stability for large $|R_r|$

3.1. *Dimensionless parameters*

For the primary transverse flow, $v(r)$, we define the angular velocity Ω and four times the vorticity ω as follows:

$$\Omega(r) = v/r, \quad \omega(r) = 2(dv/dr + v/r). \tag{3.1}$$

Then, for the type of flow being considered, the product $-\omega\Omega$ may be interpreted as a measure of the local degree of instability. In fact, Rayleigh's (1918) criterion for a fluid without viscous or magnetic stabilization states that the flow is unstable if $-\omega\Omega$ is greater than zero at any point in the fluid. In a viscous fluid, $-\omega\Omega$ must reach some positive value before instability occurs.

By dimensional analysis, we arrive at the stability parameter (a Taylor number):

$$(-\omega\Omega)\, d^4/\nu^2, \tag{3.2}$$

where d is some characteristic length of the system. The product $-\omega\Omega$ is a function of r. We use its maximum value

$$(-\omega\Omega)_m = \max_{r_1 \leqslant r \leqslant r_2} (-\omega\Omega) \tag{3.3}$$

in the definition.

With a radial Reynolds number of large magnitude,† the primary transverse profile v consists of a potential vortex throughout most of the fluid, combined with a thin boundary layer at the outlet cylinder. Such a profile should be stable except in the boundary layer. This fact, combined with the effect of the radial flow on the disturbance itself, causes the secondary flow to be confined largely to the boundary layer. The characteristic dimension of the disturbance is then proportional to the boundary-layer thickness.

In terms of the angular velocity, the primary transverse profile is given by

$$\Omega(r) = L'/r^2 + \{2M'/(R_r+2)\}r^{R_r}. \tag{3.4}$$

The first term on the right corresponds to the potential vortex. The second term produces a rapid change of velocity in the boundary layer. The radial position r_e corresponding to the edge of the boundary layer is given by

$$r_e^{R_r}/(r^{R_r})_m = \epsilon, \tag{3.5}$$

where ϵ is some rather arbitrary positive number less than 1·0, and

$$(r^{R_r})_m = \max_{r_1 \leqslant r \leqslant r_2} (r^{R_r}). \tag{3.6}$$

For outward radial flow ($R_r > 0$), we have

$$(1-\delta/r_2)^{R_r} = \epsilon, \tag{3.7}$$

where $\delta = r_2 - r_e$ is the boundary-layer thickness. For large magnitudes of R_r,

$$\delta \sim -(r_2 \ln \epsilon)/R_r \propto r_2/R_r \quad (R_r \gg 1). \tag{3.8}$$

Similarly, for inward radial flow ($R_r < 0$),

$$\delta \propto r_1/|R_r| \quad (|R_r| \gg 1). \tag{3.9}$$

The expressions r_2/R_r, $r_1/|R_r|$ are the appropriate characteristic lengths to define the dimensionless parameters for outward and inward flow, respectively. It is convenient, however, to factor the radial Reynolds number out of the resulting definitions to confine the explicit appearance of the radial flow rate to a single parameter. We define

$$T_1 = (-\omega\Omega)_m r_1^4/\nu^2, \quad \beta_1 = kr_1, \quad Q_1 = \sigma B_0^2 r_1^2/\mu \tag{3.10}$$

for inward radial flow, and

$$T_2 = (-\omega\Omega)_m r_2^4/\nu^2, \quad \beta_2 = kr_2, \quad Q_2 = \sigma B_0^2 r_2^2/\mu \tag{3.11}$$

for outward radial flow. Thus, we expect that for large $|R_r|$, the critical Taylor number (T_1 or T_2) and the critical dimensionless wave-number (β_1 or β_2) shall be proportional to $|R_r|^4$ and $|R_r|$, respectively. This is demonstrated more rigorously in the next section.

† We recall that our approximation is valid only if $P_m \ll 1$ and $P_m|R_r| \ll 1$. The magnitude $|R_r|$ can be large despite this restriction.

3.2. *Asymptotic behaviour for large* $|R_r|$

In this section we make use of the behaviour of the flow with large values of $|R_r|$ described in the above section to derive differential equations and boundary conditions which govern the asymptotic stability behaviour of the flow as $|R_r| \to \infty$. Only positive values of R_r will be considered, but similar results can be obtained when R_r is negative.

Since the disturbance resulting from instability is largely confined to an interval $r_2 - \delta \leqslant r \leqslant r_2$ which becomes relatively narrow compared to $(r_2 - r_1)$ as R_r increases, we first map that interval onto $(0, 1)$ by introducing the new independent variable

$$\zeta = \frac{r-(r_2-\delta)}{\delta} = \frac{x-(1-\delta/r_2)}{\delta/r_2}. \tag{3.12}$$

The precise value of the constant of proportionality in (3.8) is not important here. Choosing it as unity, we have

$$\zeta = \{x-(1-1/R_r)\}R_r. \tag{3.13}$$

Introducing ζ as the independent variable in (2.12), and taking the limit as $R_r \to \infty$, we obtain

$$\left.\begin{aligned} &\{(D'^2-\beta'^2)^2 - D'(D'^2-\beta'^2)+\beta'^2 Q_2/R_r^2\}\,W_1' \\ &\qquad\qquad +\beta'^2 T'\{1+\lambda\exp(\zeta-1)\}(D'^2-\beta'^2)\,W_4' = 0, \\ &\{(D'^2-\beta'^2)^2 - D'(D'^2-\beta'^2)+\beta'^2 Q_2/R_r^2\}\,W_4' - \{\exp(\zeta-1)\}\,W_1' = 0, \end{aligned}\right\} \tag{3.14}$$

where

$$\left.\begin{aligned} D' = d/d\zeta, \quad \beta' = \beta_2/R_r, \quad T' = T/R_r^4, \\ W_1'(\zeta) = R_r^{-4}\,W_1(x), \quad W_4'(\zeta) = W_4(x). \end{aligned}\right\} \tag{3.15}$$

The corresponding boundary conditions become

$$W_1', D'W_1', D'W_4', (D'^2-\beta'^2)\,W_4' = 0 \quad \text{at} \quad \zeta = 1, \tag{3.16}$$

and

$$W_1', W_4' \to 0 \quad \text{as} \quad \zeta \to -\infty. \tag{3.17}$$

The term Q_2/R_r^2 has been retained in (3.14) because its treatment depends on how the limit $R_r \to \infty$ is carried out. If the limit $R_r \to \infty$ is carried out by letting the viscosity $\mu \to 0$, then Q_2/R_r^2 remains constant and should be retained in the equations.

We shall be mainly interested, however, in the case where R_r is made large by increasing the radial flow rate with the other dimensional quantities held constant. Then Q_2 remains constant, while $Q_2/R_r^2 \to 0$ and can be dropped from the equations. The resulting critical-value problem depends only on λ. If it has a solution (β', T'), then we obtain the asymptotic stability behaviour

$$T \to R_r^4 T', \quad \beta_2 \to R_r \beta' \quad \text{as} \quad R_r \to \infty \tag{3.18}$$

independently of Q_2 and κ. Making use of the definitions of T and T_2, we find that they must have the same asymptotic behaviour; hence, as $R_r \to \infty$, we obtain

$$T_2 \propto R_r^4, \quad \beta_2 \propto R_r, \tag{3.19}$$

with the constants of proportionality again independent of Q_2 and κ.

If the terms involving Q_2/R_r^2 are dropped from (3.14), the problem becomes an ordinary hydrodynamic one; and the order of the equations can be reduced to six by introducing the new variable

$$W_2' = (D'^2 - \beta'^2)\, W_4'. \tag{3.20}$$

We then have

$$\left.\begin{aligned}\{(D'^2-\beta'^2)^2 - D'(D'^2-\beta'^2)\}\, W_1' \qquad\qquad & \\ +\beta'^2 T'\{1+\lambda \exp(\zeta-1)\}\, W_2' &= 0,\\ \{(D'^2-\beta'^2)-D'\}\, W_2' - \{\exp(\zeta-1)\}\, W_1' &= 0,\end{aligned}\right\} \tag{3.21}$$

with the boundary conditions

$$\left.\begin{aligned} W_1',\, DW_1',\, W_2' = 0 \quad &\text{at} \quad \zeta = 1,\\ W_1',\, W_2' \to 0 \quad &\text{as} \quad \zeta \to -\infty.\end{aligned}\right\} \tag{3.22}$$

For the case of a stationary wall, $\lambda = -1{\cdot}0$, (3.21) and (3.22) are identical to the equations governing the Taylor–Görtler instability of an asymptotic suction boundary layer on a curved surface, where $(1-\zeta)$ is the distance from the surface.† In fact, the case $\lambda \neq -1{\cdot}0$ can be interpreted as applying to a suction boundary on a tangentially moving curved surface, so that there is no real difference between the problems.

The terms of (3.21) involving first and third derivatives account for the effect of the primary flow normal to the wall, or suction flow, on the disturbance. They are applicable to the suction boundary layer as well as to the present problem. The authors are aware of no published results for the Taylor–Görtler problem which include the effect of suction on the disturbance. Hämmerlin (1955), however, has considered the stability of an exponential boundary-layer profile without including the suction terms. A comparison might be of interest, if only to indicate the effect of suction.

In terms of the usual Görtler parameter based on momentum thickness of the boundary layer,‡ extrapolation of the results of this paper to $R_r = \infty$ leads to the critical value 1·17, while Hämmerlin gives 0·288.

4. Discussion of results

The eigenvalue problem given by (2.12) and (2.14) has been solved numerically§ for the case of a stationary outer wall, i.e. $\lambda = -1{\cdot}0$. We consider the cases of inward and outward flow separately.

4.1. *Outward radial flow* ($R_r > 0$)

In figure 1, we present some of the typical curves showing the critical Taylor number T_2 as a function of the radial Reynolds number for several values of the radius ratio κ and Q_2. When the asymptotic form is best established, at small

† The similarity between (3.14) and the equations governing the stability of an asymptotic suction boundary layer was pointed out to the authors by one of the referees.

‡ See Hämmerlin (1955) for the definition of this parameter.

§ See appendix for a brief description of the numerical procedure.

values of κ and Q_2, the limiting slope ($\Delta \log T_2/\Delta \log R_r$) for large R_r ($\sim 10^2$)† is $+4$ as expected.

As the radial Reynolds number increases, the critical Taylor number loses its dependence on κ, again as expected.

When Q_2 is large ($\gtrsim 10^4$), the change in T_2 caused by changes in either R_r or κ decreases greatly. The asymptote for large R_r is probably the same as when $Q_2 = 0$, but it is approached much more slowly and cannot be determined with certainty from the present calculations.

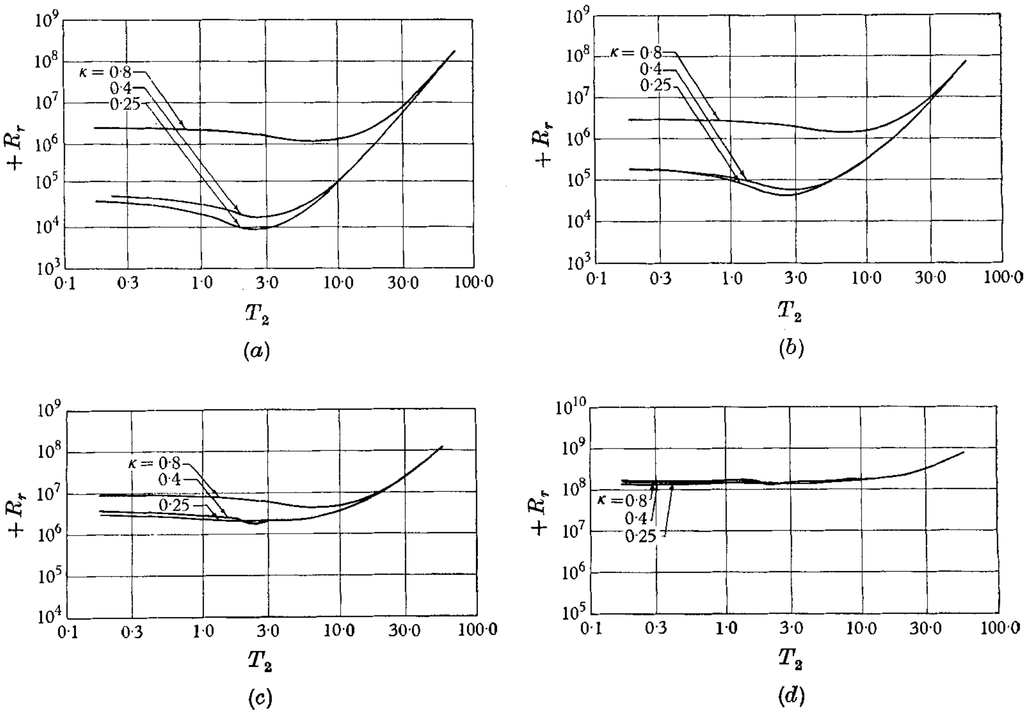

FIGURE 1. Variation of T_2 with R_r for outward radial flow, outside wall stationary. (*a*) $\lambda = -1{\cdot}0$, $Q_2 = 0$; (*b*) $\lambda = -1{\cdot}0$, $Q_2 = 100$; (*c*) $\lambda = 1{\cdot}0$, $Q_2 = 10^3$; (*d*) $\lambda = -1{\cdot}0$, $Q_3 = 10^4$.

The critical wave-number β_2, as a function of R_r, for $Q_2 = 0$, is shown in figure 2(*a*). Again, the curves become independent of κ when R_r is large and the limiting slope $+1$ is reached. When Q_2 is large, the curves for large values of κ (not shown) have the same characteristics as those for $Q_2 = 0$ except that the limiting slope of $+1$ is not yet reached in the present range of calculations.

For $Q_2 = 10^3$, 10^4, and small values of κ, discontinuities occur in the wave-number curves. One such curve ($Q_2 = 10^4$, $\kappa = 0{\cdot}25$) is shown in figure 2(*b*). The discontinuities occur when a change in the relative stability of two modes produces a sudden transition of the critical disturbance from one mode to another. A discontinuity in the wave-number curve is accompanied by an abrupt change in

† With $P_m = 10^{-6}$ or 10^{-7}, the conditions $P_m|R_r| \ll 1$ should still be satisfied.

the slope of the critical Taylor number curve. The discontinuities are believed to be a result of the restriction of these calculations to stationary critical modes. Oscillatory modes may also occur as discussed in § 5 of this paper.

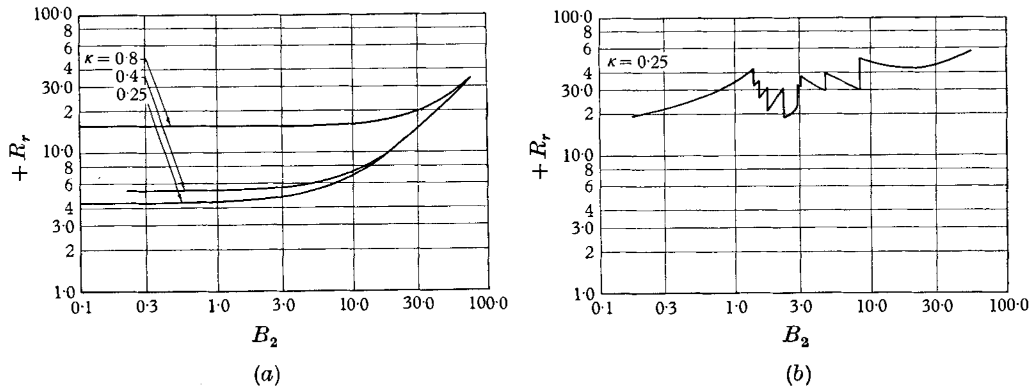

FIGURE 2. Variation of β_2 with R_r for outward radial flow, outside wall stationary. (a) $\lambda = -1·0$, $Q_2 = 0$; (b) $\lambda = -1·0$, $Q_2 = 10^4$.

4.2. *Inward radial flow* ($R_r < 0$)

The critical Taylor number curves of T_1 versus $|R_r|$ for inward radial flow are given in figure 3 for several values of κ and $Q_1 = 0, 10^3$. The qualitative behaviour is similar to that for outward flow. When $|R_r|$ is large, the value of $\Delta \log T_1/\Delta \log |R_r|$ again approaches $+4$, and the curves become independent of κ.

The corresponding critical wave-numbers for $Q_1 = 0$ are shown in figure 4(*a*). The limiting value of the slope for large $|R_r|$ is about $+1$ as expected. Similar results are obtained for $Q_1 \neq 0$. No discontinuities occur in these curves, at least up to the largest Q_1 considered. The curves for $Q_1 = 10^3$, $\kappa = 0·25$, $0·4$, however, show a rather sharp although continuous decrease in critical wave-number for $1 < |R_r| < 2$ as shown in figure 4(*b*).

4.3. *Couette flow* ($R_r = 0$)

When $R_r = 0$, the present analysis reduces to that for the stability of dissipative Couette flow under an axial magnetic field. Our results for this portion of the analysis have been reported elsewhere (Chang & Sartory 1965*a*). For the hydrodynamic case ($Q_2 = 0$), our results agreed with the experimental results of Taylor (1923), and the existing analysis of Taylor (1923), Chandrasekhar (1961), Walowit, Tsao & DiPrima (1964).

For the hydromagnetic case ($Q_2 \neq 0$), our results for perfectly conducting walls agreed quite well with the calculated results of Chandrasekhar (1961) for $Q_2 \gtrsim 10^3$. Due to the short-circuiting of current through the perfectly conducting walls, the stationary convective wave cells did not elongate indefinitely with the increase of Q_2, however; and a viscosity-independent asymptote of $T \propto Q_2^{\frac{3}{2}}$ as $Q_2 \to \infty$ was observed.

Actually, a complicated phenomenon of transition to oscillatory modes of instability occurs at high values of Q_2 for perfectly conducting walls. When the oscillatory modes are admitted, the convective cells elongate with Q_2 as expected and the viscosity-dependent asymptote of $T \propto Q_2$ as predicted by Chandrasekhar is obtained. A detailed description of this behaviour was given by Chang & Sartory (1965*b*)

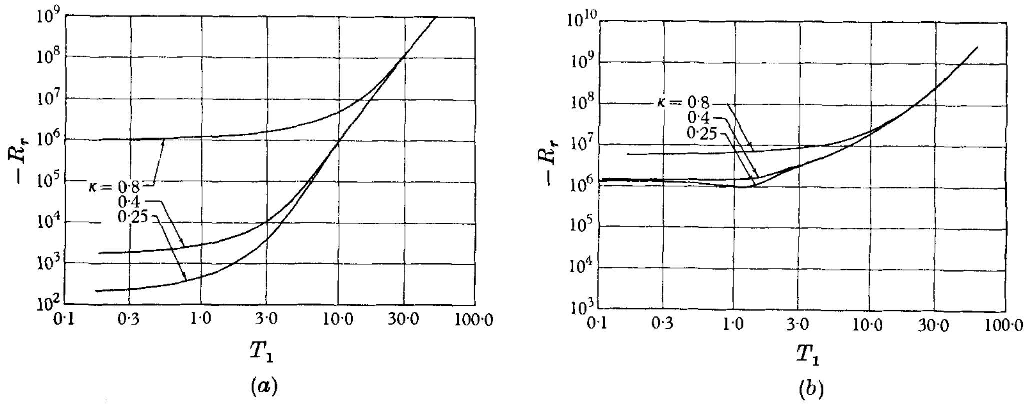

FIGURE 3. Variation of T_1 with R_r for inward radial flow, outside wall stationary. (*a*) $\lambda = -1{\cdot}0$, $Q_1 = 0$; (*b*) $\lambda = -1{\cdot}0$, $Q_1 = 10^3$.

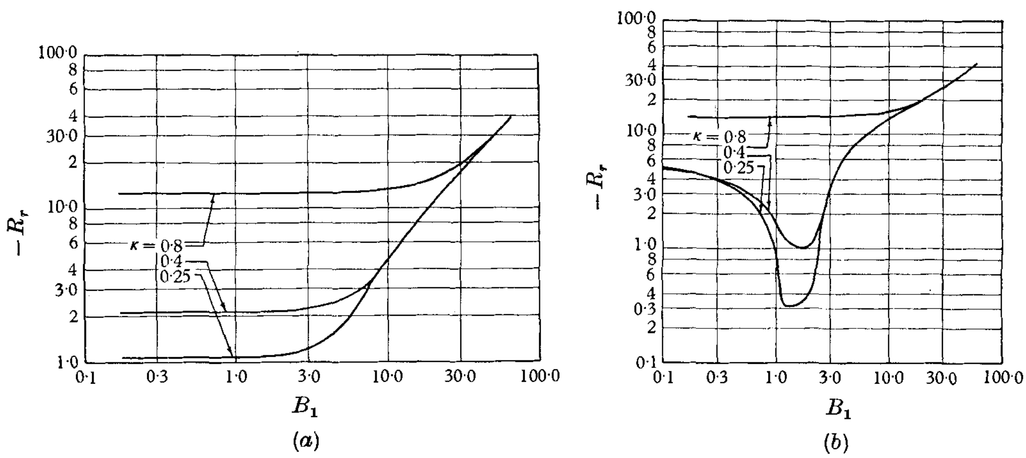

FIGURE 4. Variation of β_1 with R_r for inward radial flow, outside wall stationary. (*a*) $\lambda = -1{\cdot}0$, $Q_1 = 0$; (*b*) $\lambda = -1{\cdot}0$, $Q_1 = 10^3$.

5. Oscillatory disturbances

In this paper, calculations are restricted to stability with respect to stationary axisymmetric disturbances. As described in § 4.3, for the case of hydromagnetic stability of Couette flow between non-permeable perfectly conducting cylinders, the authors have reported earlier (1965*b*) that oscillatory modes appear and become more critical than stationary modes for large values of Q_2. Since the present results reduce to the non-permeable wall case as $R_r \to 0$, it is certainly to be expected that oscillatory critical disturbances will also occur with permeable walls, at least when Q_1 (or Q_2) is large and $|R_r|$ is small. While the effect of radial flow on the oscillatory modes is not yet completely understood, preliminary calculations indicate that as $|R_r| \to \infty$, the stationary modes reported in this paper are more critical.

If the results of this paper are extended to include oscillatory as well as stationary axisymmetric critical modes, it is expected that:

(*a*) The $|R_r| \to \infty$ asymptotic behaviour will not change.

(*b*) The graphs of critical Taylor and wave-number for small values of Q_1 or Q_2 will not change.

(*c*) In the graphs of Taylor and wave-number for large values of Q_1 or Q_2, that part of the curves corresponding to small $|R_r|$ will be altered. In particular, the critical Taylor number will be lowered, and the complicated series of discontinuities shown in figure 2(*b*), for example, will be altered or eliminated.

The stability of oscillatory disturbances will be considered in detail in part 2 of this series.

6. Conclusions

(*a*) The asymptotic stability analysis indicates that as $|R_r|$ becomes large, the critical Taylor number T_1 (or T_2) becomes proportional to R_r^4 and the critical wave-number β_1 (or β_2) becomes proportional to R_r.

The asymptotes are independent of κ.

If the limit $|R_r| \to \infty$ is taken with a constant value of Q_1 (or Q_2) (this corresponds to increasing the radial flow rate while holding the other dimensional parameters constant) then the asymptotes are also independent of Q_1 (or Q_2).

(*b*) Results obtained from numerical solution of the stability equations, for all values of Q_1 (or Q_2) considered, indicate that the critical Taylor and wave-numbers become independent of κ for large values of $|R_r|$ as expected from the asymptotic analysis.

Numerical results also verify the expected asymptotic behaviour T_1 (or T_2) $\propto R_r^4$, β_1 (or β_2) $\propto R_r$, when Q_1 (or Q_2) is small. As Q_1 (or Q_2) is increased, the asymptote is approached more slowly. It is believed that the same asymptotes apply also for large values of Q_1 (or Q_2), but the range of R_r covered in the calculations is not sufficient for verification.†

It is possible to represent the Taylor number for critical stationary modes as a function of R_r, Q, and κ on a single three-dimensional graph. We define

$$T_* = (-\omega\Omega)_m d_*^4/\nu^2, \quad Q_* = \sigma B_0^2 d_*^4/\mu \tag{6.1}$$

with

$$d_* = (e^{R_r} r_2 + e^{-R_r} r_1)/(e^{R_r} + e^{-R_r}). \tag{6.2}$$

Since $d_* \to r_1$ (or r_2) as $R_r \to -\infty$ (or $+\infty$), the asymptotic behaviour of T_* for large $|R_r|$ must be

$$T_* \propto |R_r|^4 \quad \text{as} \quad |R_r| \to \infty. \tag{6.3}$$

We now map the critical values of T_* as a function of R_r and Q_* for each κ. Since the critical T_* becomes independent of κ for large values of $|R_r|$ or Q_*, the resulting graph must have a dish-like shape as shown qualitatively in figure 5.

(*c*) With outward radial flow, when Q_2 is large and R_r is small, a series of discontinuities occurs in the graphs of critical wave-number.

† However, note that if Q_1 (or Q_2) is very large, the value of $|R_r|$ required to reach the asymptote might be so great that the restriction $P_m|R_r| \ll 1$ is violated.

With inward radial flow, when Q_1 is large and $|R_r|$ is small, a rather sharp, but continuous decrease in wave-number occurs with $1 \leqslant |R_r| \leqslant 2$.

(*d*) It is known from earlier work (Chang & Sartory 1965*b*) on the hydromagnetic stability of Couette flow between conducting non-permeable walls, that

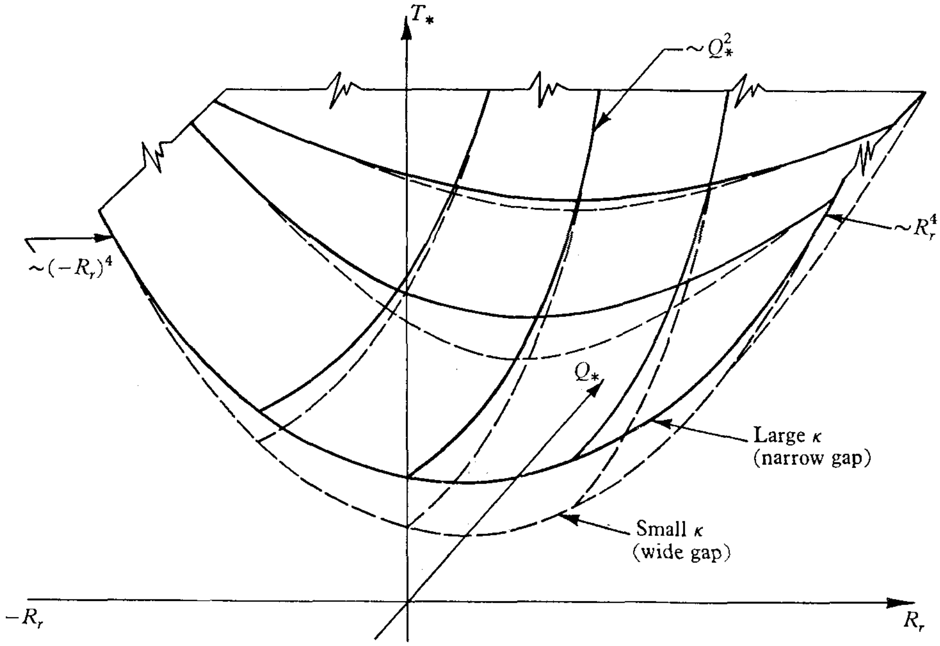

FIGURE 5. Qualitative sketch of T_* as a function of R_r and Q_* for stationary critical modes, outside wall stationary.

for large values of the Hartman number, oscillatory modes are more unstable than the stationary disturbances considered in this paper. This result will certainly apply also to flow between permeable walls when the radial Reynolds number is small, but large values of $|R_r|$ are expected to inhibit the oscillatory disturbances. Of the conclusions listed above, only (*c*) should change if the present results are extended to include oscillatory as well as stationary axisymmetric critical disturbances.

This research was sponsored by the U.S. Atomic Energy Commission under contract with the Union Carbide Corporation. A summary of this paper was presented at the Seventh British Theoretical Mechanics Colloquium, Leeds, England, April 1965. The authors are grateful to Dr J. J. Keyes, for his critical and encouraging comments. One of the authors (T. S. Chang) wishes to thank Prof. G. K. Batchelor for the hospitality extended to him during his visit at the Department of Applied Mathematics and Theoretical Physics, University of Cambridge.

Note added in proof. The authors are grateful to Drs S. K. Datta and E. R. Krueger for a preprint of their paper, *On the stability of Couette flow with suction and injection*. Their narrow-gap, nonmagnetic results seem to be consistent with the wide-gap results given in this paper for $B_0 \rightarrow 0$.

Appendix

Numerical method

The numerical procedure used to obtain the results of this paper is essentially the same as that described by the authors in an earlier paper (Chang & Sartory 1965*c*). The interval $\kappa \leqslant x \leqslant 1$ is divided into a mesh of equal subintervals, and the differential equations and boundary conditions are replaced by a homogeneous system of linear algebraic equations by approximating the derivatives with appropriate finite-difference expressions. The condition that the resulting equations have a solution is that the determinant of their coefficients vanishes. The determinant is evaluated numerically by Gaussian elimination. The equation

$$\det(\beta_2, T) = 0 \tag{A 1}$$

defines implicitly the curve of neutral stability. At a (relative) minimum of the neutral curve

$$\frac{dT}{d\beta_2} \quad \text{or} \quad \frac{\partial \det(\beta_2, T)}{\partial \beta_2} = 0. \tag{A 2}$$

The simultaneous equations (A 1) and (A 2) are solved by Newton's method, again approximating the required derivatives of the determinant by appropriate difference expressions, e.g.

$$\frac{\partial \det(\beta_2, T)}{\partial \beta_2} \approx \frac{\det\{\beta_2(1+\Delta), T\} - \det\{\beta_2(1-\Delta), T\}}{2\beta_2 \Delta}. \tag{A 3}$$

To insure adequate accuracy of these approximations, the increment Δ is decreased by 50 % in each of the final steps of Newton's method. Convergence of Newton's method is assumed when the values of both β_2 and T obtained in two successive steps differ by less than 0·01 %.†

The entire calculation outlined above is then repeated using twice the number of mesh points for the approximation of the differential equations. The number of mesh points is assumed to be adequate when the values of β_2 and T obtained with two successive mesh sizes differ by less than 0·5 %.

The starting values of β_2 and T required by Newton's method are usually supplied by extrapolation of known results. Occasionally, however, graphs of the determinant or of the Taylor number for neutral stability versus wave-number are required. The neutral Taylor number, when needed, is calculated essentially as described above, except that equation (A 2) is not used, and β_2 is treated as a parameter.

It was found during the calculations that the neutral curves of T versus β_2, at large values of Q_2, possess several relative minima, any one of which can be the absolute minimum or critical point. Such a neutral curve is shown in figure 6. As the radial Reynolds number is varied, the minima are affected by differing amounts, and transitions between minima occur, leading to abrupt changes in critical wave-number as shown in figure 2 (*b*).

Figure 6 shows the neutral values of (T_2, β_2) for the first ten normal modes. The curves for higher modes resemble the two curves to the left of the graph, and

† Occasionally, when this criterion is not met after six steps of Newton's method, a weaker acceptance criterion of 0·1 % is substituted.

lie above them. If the value of Q_2 were increased, the two curves to the left would close at some small value of the wave-number to form a single loop, and further increases in Q_2 would cause the loop to recede to the right until it resembled the other four loops shown. Then the minimum of the next mode would be exposed, and would have to be included in the calculation.

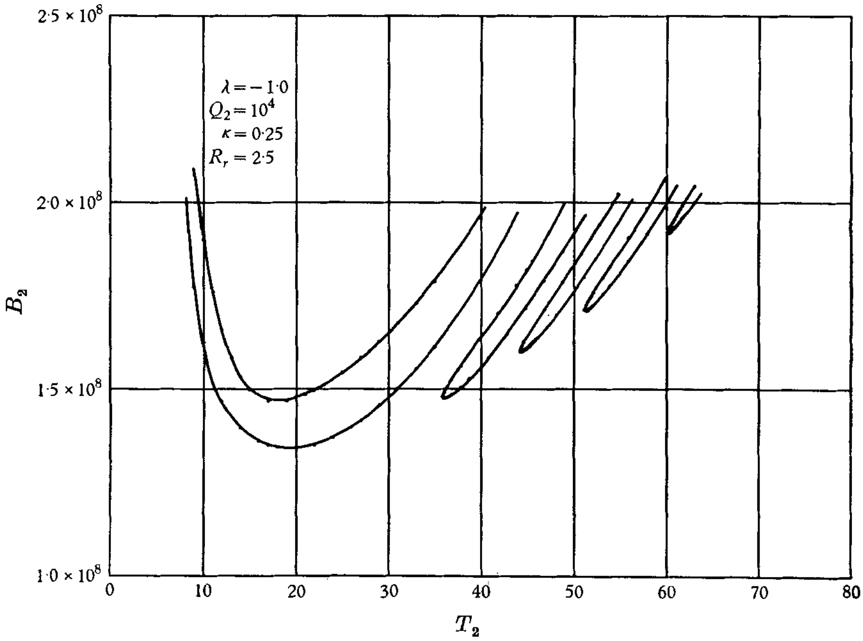

FIGURE 6. Curves of T_2 versus β_2 for stationary behaviour of the first ten normal modes. $\lambda = -1{\cdot}0$, $Q_2 = 10^4$, $\kappa = 0{\cdot}25$, $R_r = 2{\cdot}5$.

The calculation procedure which has been described so far is not capable of following transitions between the competing minima illustrated in figure 6. To detect transitions, the following checking process is used.

The function $\det(\beta_2, T)$ is normalized so that $\det(\beta_2, 0) > 0$. Then, referring to figure 6, the value of the determinant is negative within the finger-like loops on the right and in the strip between the two curves to the left of the graph, and positive elsewhere. After a relative critical point (relative minimum on the T versus β_2 curve) is calculated, the resulting Taylor number is decremented by 1 % and held constant while the determinant is evaluated at a series of 50 or 100 values of β_2 distributed over the interval in which minima are anticipated, say 1·0 to 100. If a negative value of the determinant is detected, indicating that another loop of the curve extends below the calculated relative critical point, a search is conducted for the new minimum. The value of the radial Reynolds number at which a transition occurs is given only approximately by the checking process, and a graph of Taylor number versus radial Reynolds number for the two minima in question is used to determine it more precisely.

To provide a quantitative check on the present calculations, we include in table 1 a comparison of some results obtained by the present method with results given in table 1 of Walowit, Tsao & DiPrima (1964). In the nomenclature of the present paper, they tabulate $(1-\kappa)\beta_2$ and $T_2(1-\kappa)^4\kappa^2$ for several values of κ. The greatest difference is about 1 % and occurs at the smallest value of κ.

	Present work		Walowit, Tsao & DiPrima (1964)	
κ	$(1-\kappa)\beta_2$	$(1-\kappa)^4T_2\kappa^2$	$(1-\kappa)\beta_2$	$(1-\kappa)^4T_2\kappa^2$
0·8	3·14	2551·7	3·13	2553·4
0·6	3·148	1851·5	3·15	1851·5
0·4	3·186	1279·2	3·17	1279·1
0·1	3·339	645·6	3·30	650·0

TABLE 1. Comparison of critical parameters; no radial flow, no magnetic field, stationary outer wall.

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